STABILITY IN L^1 OF CIRCULAR VORTEX PATCHES

THOMAS C. SIDERIS AND LUIS VEGA

ABSTRACT. The motion of incompressible and ideal fluids is studied in the plane. The stability in L^1 of circular vortex patches is established among the class of all bounded vortex patches of equal strength.

For planar incompressible and ideal fluid flow, the theory of Yudovich [9] establishes global well-posedness of the initial value problem with initial vorticity in $L^1(\mathbb{R}^2) \cap L^{\infty}(\mathbb{R}^2)$. Because vorticity is transported in 2d, it remains constant along particle trajectories. If Φ_t is the flow map, then the vorticity is given by $\omega(t, \Phi_t(y)) = \omega(0, y)$, for all t > 0 and $y \in \mathbb{R}^2$. When the initial vorticity is a patch of unit strength, represented by the indicator (characteristic) function I_{Ω_0} of a bounded open set $\Omega_0 \subset \mathbb{R}^2$, the resulting vorticity is I_{Ω_t} , with $\Omega_t = \Phi_t(\Omega_0)$. In the special case when Ω_0 is equal to a ball B, the patch is stationary, $\Phi_t(B) = B$, for all t > 0. Theorem 3, our main result, gives the stability in $L^1(\mathbb{R}^2)$ of any circular patch within the class of all bounded vortex patches of equal strength. No restriction is placed on the L^1 distance of the perturbation to the ball, and the flow region is not limited to a bounded domain, but rather is the entire space \mathbb{R}^2 .

Wan and Pulvirenti [8] were the first to study stability of vortex patches in L^1 . They considered the case where the flow was contained in a bounded region, although for the stability of circular patches they mention that this assumption can be removed. Their key estimate, (**J**), shows that the total angular momenta of the patches can be used to control the L^1 difference between an arbitrary patch and a circular patch of the same total mass. They allow the strengths of the patches differ, in which case the two patches are assumed to be close in L^1 . Our generalization of their inequality, given in Lemma 2, estimates the L^1 distance of an arbitrary patch to a circular patch, when both

Date: May 28, 2009.

²⁰⁰⁰ Mathematics Subject Classification. Primary 35Q35, secondary 76B47.

T.C.S. was supported by a grant from the National Science Foundation.

 $^{{\}rm L.V.}$ was supported by a grant from the Ministerio de Educación y Ciencia, MTM2007-62186.

The authors thank the anonymous referee for helpful comments.

patches have equal strength. Stability in L^1 , given in Theorem 3, follows immediately. Weaker stability results were given by Saffman [7] and Dritschel [4]. Dritschel controls the measure of the symmetric difference of two patches through a convenient integral, and this idea is incorporated into our argument in Lemma 1.

Stability in L^1 does not imply that the boundaries of the two patches remain close in any metric. Indeed, numerical simulations give strong evidence of fingering and filamentation, see [1, 3]. Spreading of vorticity may also occur. The best upper bound for the growth rate of the patch diameter is $\mathcal{O}(t \log t)^{1/4}$ given in [5], see also [6]. Nevertheless, in spite of the fact that the patch geometry may be complicated, smoothness of smooth patch boundaries persists for all time, see [2].

For any bounded open set $A \subset \mathbb{R}^2$, denote its mass, momentum, and angular momentum by

$$|A| = \int_A dx$$
, $M(A) = \int_A x \, dx$, and $i(A) = \int_A |x|^2 dx$,

respectively. Our arguments depend heavily upon the fact that these three quantities are conserved in time when $A = \Omega_t$ is a patch moving with the flow.

Lemma 1. If $A \subset \mathbb{R}^2$ is any bounded open set, then

$$i(A) - \frac{|A|^2}{2\pi} - \frac{|M(A)|^2}{|A|} \ge 0.$$

Equality holds if and only if the set A is a ball.

Proof. For any ball $B_r(x_0) = \{x \in \mathbb{R}^2 : |x - x_0| < r\}$, introduce the quantity

(1)
$$Q = Q(A; B_r(x_0)) = \int_{A \triangle B_r(x_0)} ||x - x_0|^2 - r^2| dx,$$

in which $A \triangle B_r(x_0) = (A \backslash B_r(x_0)) \cup (B_r(x_0) \backslash A)$ denotes the symmetric difference. Note that $Q \ge 0$ and Q = 0 if and only if $A = B_r(x_0)$.

The quantity Q can also be written as

$$Q = \int_{A} (|x - x_0|^2 - r^2) dx + \int_{B_r(x_0)} (r^2 - |x - x_0|^2) dx,$$

since the portions of these two integrals over the set $A \cap B_r(x_0)$ cancel. Now, we can expand the first integral in Q and compute the second to obtain

$$Q = i(A) - 2x_0 \cdot M(A) + (|x_0|^2 - r^2)|A| + \frac{\pi}{2}r^4.$$

A rearrangement of terms gives

(2)
$$Q = i(A) - \frac{|A|^2}{2\pi} - \frac{|M(A)|^2}{|A|} + \frac{1}{2\pi} \left(\pi r^2 - |A| \right)^2 + |A| \left| x_0 - \frac{M(A)}{|A|} \right|^2.$$

This last expression is minimized by choosing $B_r(x_0)$ with the same mass and center of mass as A:

$$|B_r(x_0)| = \pi r^2 = |A|$$
 and $x_0 = \frac{M(A)}{|A|}$.

With this choice, the Lemma now follows.

Lemma 2. If $B = B_r(0)$, then for any bounded open set A,

$$||I_A - I_B||_{L^1}^2 \le 4\pi \ Q(A; B)$$

in which Q(A; B) is defined by (1). Equality holds if and only if

$$(3) A = B_a(0) \cup [B_b(0) \setminus B_r(0)],$$

with a < r < b and $r^2 - a^2 = b^2 - r^2$.

Proof. Using the identity (2) and then Lemma 1, we have for any bounded open set A',

(4)
$$(|A'| - |B|)^2 = (|A'| - \pi r^2)^2 \le 2\pi \ Q(A'; B),$$

with equality if and only if A' is a ball centered at the origin.

Next, we note that

$$||I_A - I_B||_{L^1}^2 = |A\Delta B|^2$$

$$= (|A \setminus B| + |B \setminus A|)^2$$

$$\leq 2|A \setminus B|^2 + 2|B \setminus A|^2$$

$$= 2(|A \cup B| - |B|)^2 + 2(|A \cap B| - |B|)^2,$$

with equality if and only if $|A \setminus B| = |B \setminus A|$.

Application of (4) with $A' = A \cup B$ and $A' = A \cap B$ yields

$$2(|A \cup B| - |B|)^2 + 2(|A \cap B| - |B|)^2$$

$$\leq 4\pi \left[Q(A \cup B; B) + Q(A \cap B; B) \right] = 4\pi \ Q(A; B),$$

with equality if and only if $A \cup B$ and $A \cap B$ are balls centered at the origin.

This establishes the desired inequality. The argument also shows that equality holds if and only if $A \cup B = B_b(0)$, $A \cap B = B_a(0)$, with a < r < b, and

$$|B_b(0) \setminus B| = |A \setminus B| = |B \setminus A| = |B \setminus B_a(0)|,$$

which gives (3).

Theorem 3. Let $B = B_r(0)$. Then for any bounded open set $\Omega_0 \subset \mathbb{R}^2$, we have that

$$||I_{\Omega_t} - I_B||_{L^1}^2 \le 4\pi \sup_{\Omega_0 \triangle B} ||x|^2 - r^2| ||I_{\Omega_0} - I_B||_{L^1},$$

for all t > 0.

Proof. The identity (2) shows that the quantity $Q(\Omega_t; B)$ depends only on conserved quantities, and it is therefore also conserved. In other words, we have $Q(\Omega_t; B) = Q(\Omega_0; B)$, for all t > 0. Thus, the result follows from Lemma 2 and the fact that

$$Q(\Omega_0; B) \le \sup_{\Omega_0 \triangle B} ||x|^2 - r^2| ||\Omega_0 \triangle B| = \sup_{\Omega_0 \triangle B} ||x|^2 - r^2| ||I_{\Omega_0} - I_B||_{L^1}.$$

References

- [1] Buttke, T.F. A fast adaptive vortex method for patches of constant vorticity in two dimensions. J. Comput. Phys. 89 (1990), no. 1, 161–186.
- [2] Chemin, J.-Y. Sur le mouvement des particules d'un fluide parfait, incompressible, bidimensionnel. Inven. Math. 103 (1991), 599–629.
- [3] Deem, G.S. and N.J. Zabusky. Vortex waves: stationary "V states", interactions, recurrence, and breaking. Phys. Rev. Lett. 41 (1978), no. 13, 859–862.
- [4] Dritschel, D.G. Nonlinear stability bounds for inviscid, two-dimensional, parallel or circular flows with monotonic vorticity, and the analogous three-dimensional quasi-geostrophic flows. J. Fluid Mech. 191 (1988), 575-581.
- [5] Iftimie, D., T.C. Sideris, and P. Gamblin. On the evolution of compactly supported planar vorticity. Comm. Partial Differential Equations **24** (1999), no. 9-10, 1709–1730.
- [6] Marchioro, C. Bounds on the growth of the support of a vortex patch. Comm Math. Phys. **164** (1994), 507–524.
- [7] Saffman, P.G. *Vortex dynanics*. Cambridge Monographs on Mechanics and Applied Mathematics. Cambridge University Press. New York: 1997. (cf. p. 167).
- [8] Wan, Y.H. and M. Pulvirenti. *Nonlinear stability of circular vortex patches*. Comm. Math. Phys. **99** (1985), no. 3, 435–450.
- [9] Yudovich, V.I. Non-stationary flow of an ideal incompressible liquid. Zh. Vych. Mat. 3 (1963), no. 6. 1032–1066.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, SANTA BARBARA, CA 93106, USA

E-mail address: sideris@math.ucsb.edu

Departamento de Matemáticas, Universidad del País Vasco, Apartado 644, 48080 Bilbao, Spain,

E-mail address: luis.vega@ehu.es